The Muffin Problem

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How it Began

A Recreational Math Conference
(Gathering for Gardner)
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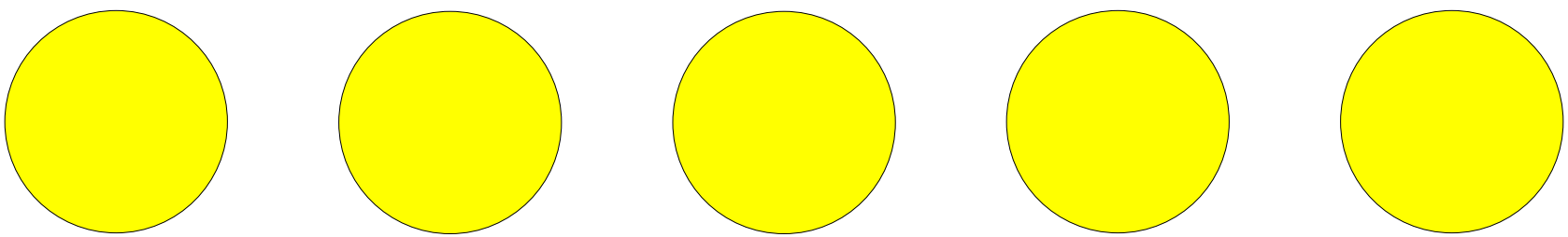
I found a pamphlet:

The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

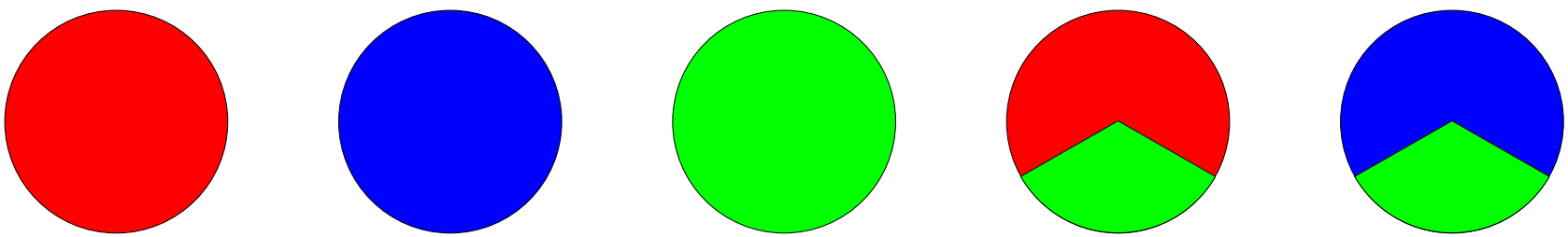
How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?



Five Muffins, Three Students, Proc by Picture

Person	Color	What they Get
Alice	**RED**	$1 + \frac{2}{3} = \frac{5}{3}$
Bob	**BLUE**	$1 + \frac{2}{3} = \frac{5}{3}$
Carol	GREEN	$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$

Smallest Piece: $\frac{1}{3}$



Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

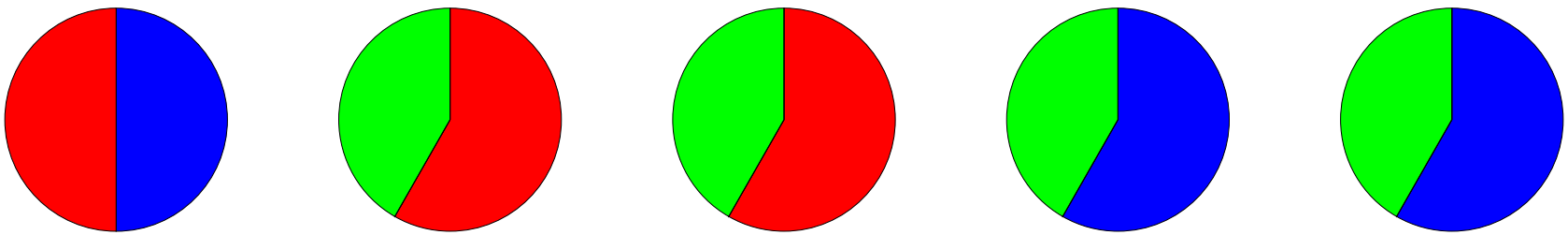
Work on it with your neighbor

Five Muffins, Three People–Proc by Picture

YES WE CAN!

Person	Color	What they Get
Alice	**RED**	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Bob	**BLUE**	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Carol	GREEN	$\frac{5}{12} + \frac{5}{12} + \frac{5}{12} + \frac{5}{12}$

Smallest Piece: $\frac{5}{12}$



Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

Work on it with your neighbor

Five Muffins, Three People–Can't Do Better Than $\frac{5}{12}$

NO WE CAN'T!

There is a procedure for 5 muffins,3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N. We want $N \leq \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases.
(**Henceforth:** All muffins are cut into ≥ 2 pieces.)

Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$.
(**Henceforth:** All muffins are cut into 2 pieces.)

Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets ≥ 4 pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12} \qquad \text{Great to see } \frac{5}{12}$$

3 Muffins, 5 Students?

Clearly can do with smallest pice $\frac{1}{5}$.

Work on it with your neighbor

3 Muffins, 5 students, Smallest piece *frac*14

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

3 Muffins, 5 students, Smallest piece *frac*14

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

Can we do better?

Work on it with your neighbor

3 Muffins, 5 Students—Can't Do Better Than $\frac{1}{4}$?

VOTE: YES, NO, UNKNOWN

3 Muffins, 5 Students—Can't Do Better Than $\frac{1}{4}$?

VOTE: YES, NO, UNKNOWN

NO WE CAN'T!

There is a procedure for 3 muffins,5 students where each student gets $\frac{3}{5}$ muffins, smallest piece N. We want $N \leq \frac{1}{4}$.

Case 0: Alice gets 1 piece of size $\frac{3}{5}$. Look at the rest of that muffin which totals to $\frac{2}{5}$. (1) That piece is cut. Have piece $\leq \frac{2}{5} \times \frac{1}{2} = \frac{1}{5}$, OR (2) That piece uncut. So someone gets a $\frac{2}{5}$-piece. Must also get a $\frac{1}{5}$ piece.
(**Henceforth:** All people get ≥ 2 pieces.)

Case 1: Alice gets ≥ 3 pieces. Then $N \leq \frac{3}{5} \times \frac{1}{3} = \frac{1}{5}$.
(**Henceforth:** Everyone gets 2 pieces.)

Case 2: Everyone gets 2 pieces. 10 pieces, 3 muffins:
Some muffin gets ≥ 4 pieces. So some piece is $\leq \frac{1}{4}$.

Three-Five and Five-Three

Five Muffins, Three Students:

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

Three-Five and Five-Three

Five Muffins, Three Students:

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

Three Students, Five Students:

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

Work out More for Three Students

Work out with your neighbor

4 muffins 3 studetns

5 muffins 3 students

6 muffins 3 student

etc.